

A NOTE ON REDUCTIONS OF 2-DIMENSIONAL CRYSTALLINE GALOIS REPRESENTATIONS

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ABSTRACT. Let p be an odd prime number, K_f the finite unramified extension of \mathbb{Q}_p of degree f , and G_{K_f} its absolute Galois group. We construct analytic families of étale (φ, Γ) -modules which give rise to some families of 2-dimensional crystalline representations of G_{K_f} with largest Hodge-Tate weight at least p . As an application, we prove that the modulo p reductions of the members of each such family (with respect to appropriately chosen Galois-stable lattices) are constant.

INTRODUCTION

Two-dimensional crystalline representations of $G_{\mathbb{Q}_p} := \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ arise from classical newforms of level prime to p . Modulo p reductions of such representations with Hodge-Tate weights in the range $[0; p]$ mattered for the proof of Serre's modularity conjecture by Khare and Wintenberger. The reductions of all irreducible 2-dimensional crystalline representations of $G_{\mathbb{Q}_p}$ with Hodge-Tate weights in this range were computed by Berger-Li-Zhu [4], who extended previous results of Deligne, Edixhoven, Fontaine and Serre.

Serre's modularity conjecture has been recently generalized by Buzzard, Diamond and Jarvis [5] for irreducible totally odd 2-dimensional $\overline{\mathbb{F}_p}$ -representations of the absolute Galois group of any totally real field unramified at p . Two-dimensional crystalline representations of $G_{K_f} := \text{Gal}(\overline{\mathbb{Q}_p}/K_f)$ arise naturally in the context of the BDJ conjecture, and their modulo p reductions are important for the formulation of the weight part of this conjecture (see [5, §3]). For Hodge-Tate weights in the range $[0; p-1]$, the modulo p reductions of the irreducible 2-dimensional crystalline representations of G_{K_f} can be easily computed using Fontaine-Laffaille theory. For arbitrary Hodge-Tate weights, semisimplified modulo p reductions of certain families of 2-dimensional crystalline representations of G_{K_f} were computed in [8], extending the constructions in [4] from \mathbb{Q}_p to K_f .

More precisely, for any 2-dimensional crystalline representation V of G_{K_f} with Hodge-Tate type $\text{HT}_V(\tau) = \{0, -k_i\}$, where the k_i are nonnegative integers, which is up to unramified twist either irreducible and induced from a crystalline character of $G_{K_{2f}}$ or a split-reducible and non-ordinary, we constructed an infinite family $\mathcal{F}(V)$ of 2-dimensional crystalline representations of G_{K_f} with the following properties:

- (1) $V \in \mathcal{F}(V)$;
- (2) The members of $\mathcal{F}(V)$ have Hodge-Tate type $\text{HT}_V(\tau)$;
- (3) The members of $\mathcal{F}(V)$ have the same modulo p reductions with respect to appropriately chosen Galois-stable lattices.

The members of $\mathcal{F}(V)$ were described in terms of their corresponding by the Colmez-Fontaine theorem (see [6, Théorème A]) weakly admissible filtered φ -modules. For each family $\mathcal{F}(V)$, the semisimplification $\overline{\mathcal{F}(V)}^{ss}$ of the common reduction is independent of choices of lattices, and was

explicitly computed in ([8, Theorems 1.5 & 1.7]). Recall that if V is reducible then $\mathcal{F}(V)$ can contain both irreducible and reducible representations (see [8], comments after Theorem 1.7).

The modulo p reductions of all 2-dimensional crystalline $\overline{\mathbb{Q}_p}$ -representations of G_{K_f} with Hodge-Tate weights in the range $[0; p]$ in terms of the corresponding by the Colmez-Fontaine weakly admissible filtered φ -modules are currently unknown. The goal of this paper is to enlarge the families $\mathcal{F}(V)$ to families of 2-dimensional crystalline $\overline{\mathbb{Q}_p}$ -representations of the same Hodge-Tate type and with constant modulo p reductions with respect to appropriately chosen Galois-stable lattices, under the assumption that the largest Hodge-Tate weight is at least $p \neq 2$. The proof rests on Wach module constructions and makes use of the constructions in [8] and an idea of Berger ([3, §10.3]).

1. DESCRIPTION OF THE FAMILIES

Throughout this paper p will be a fixed odd integer prime, $K_f = \mathbb{Q}_{p^f}$ the finite unramified extension of \mathbb{Q}_p of degree f , and E a finite extension of K_f with ring of integers \mathcal{O}_E , maximal ideal \mathfrak{m}_E , and residue field k_E . When the degree of K_f plays no role we simply write K . We denote by σ_K the absolute Frobenius of K ; we fix once and for all a distinguished embedding $K \xrightarrow{\tau_0} E$ and we let $\tau_j = \tau_0 \circ \sigma_K^j$ for all $j = 0, 1, \dots, f-1$. We fix the f -tuple of embeddings $|\tau| := (\tau_0, \tau_1, \dots, \tau_{f-1})$ and we denote $E^{|\tau|} := \prod_{\tau: K \hookrightarrow E} E$, with the embeddings ordered as above. For the language of crystalline representations see [9].

Notation 1.1. Let k_i be fixed nonnegative integers which we call weights. Assume that after ordering them and omitting possibly repeated weights we get $w_0 < w_1 < \dots < w_{t-1}$, where w_0 is the smallest weight, w_1 the second smallest weight, ..., and w_{t-1} is the largest weight for some $1 \leq t \leq f$. The largest weight w_{t-1} will be usually denoted by k and throughout the paper we assume that $k \geq p$. For convenience we define $w_{-1} = 0$. Let $I_0 := \{0, 1, \dots, f-1\}$; for $j = 1, 2, \dots, t-1$ let $I_j := \{i \in I_0 : k_i > w_{j-1}\}$ and let $I_t = \emptyset$. For each subset $J \subset I_0$ we write $f_J := \sum_{i \in J} e_i$ and $E^{|\tau_J|} := f_J \cdot E^{|\tau|}$. The sets $E^{|\tau_{I_j}|}$ are obtained as follows: $E^{|\tau_{I_0}|}$ is the Cartesian product E^f . Starting with $E^{|\tau_{I_0}|}$, we obtain $E^{|\tau_{I_1}|}$ by killing the coordinates where the smallest weight occurs. We obtain $E^{|\tau_{I_2}|}$ by further killing the coordinates where the second smallest weight w_1 occurs and so on.

We first recall the construction of the families $\mathcal{F}(V)$ in [8]. For $i = 0, 1, \dots, f-1$, let χ_i be a crystalline E -character of G_{K_f} with Hodge-Tate type $\text{HT}_{\chi_i}(\tau_{i+1}) = \{-1\}$, where indices are viewed modulo f . Let $\{\ell_j\}_{0 \leq j \leq 2f-1}$ be integers such that $\{\ell_i, \ell_{f+i}\} = \{0, k_i\}$ for all $i = 0, 1, \dots, f-1$. Up to unramified twist, any irreducible 2-dimensional crystalline representation V of G_{K_f} of Hodge-Tate type $\text{HT}_V(\tau) = \{0, -k_i\}$ which is induced from a crystalline character of $G_{K_{2f}}$ has the form $V = \text{Ind}_{K_{2f}}^{K_f}(\chi_{\vec{\ell}})$, where $\chi_{\vec{\ell}} = \chi_0^{\ell_1} \cdot \chi_1^{\ell_2} \cdots \chi_{2f-2}^{\ell_{2f-1}} \cdot \chi_{2f-1}^{\ell_0}$ (cf. [8, Theorem 1.3]). Any split-reducible non-ordinary 2-dimensional crystalline representation V of G_{K_f} of Hodge-Tate type $\text{HT}_V(\tau) = \{0, -k_i\}$ is up to unramified twist of the form

$$V = \eta \cdot \chi_0^{\ell_1} \cdot \chi_1^{\ell_2} \cdots \chi_{f-2}^{\ell_{f-1}} \cdot \chi_{f-1}^{\ell_0} \oplus \chi_0^{\ell_{1+f}} \cdot \chi_1^{\ell_{2+f}} \cdots \chi_{2f-2}^{\ell_{2f-1}} \cdot \chi_{2f-1}^{\ell_f},$$

where η is an unramified character, with both vectors $(\ell_0, \ell_1, \dots, \ell_{f-1})$ and $(\ell_f, \ell_{f+1}, \dots, \ell_{2f-1})$ nonzero (cf. [8, Theorem 1.7]). Fix a representation V as above. Let $\{X_i\}_{1 \leq i \leq f}$ be a set of indeterminates and let $P_i(X_i) \in M_2(\mathcal{O}_E[[X_i]])$ be a matrix of one of the following four types:

$$t_1: \begin{pmatrix} p^{k_i} & 0 \\ X_i & 1 \end{pmatrix}, \quad t_2: \begin{pmatrix} X_i & 1 \\ p^{k_i} & 0 \end{pmatrix}, \quad t_3: \begin{pmatrix} 1 & X_i \\ 0 & p^{k_i} \end{pmatrix}, \quad t_4: \begin{pmatrix} 0 & p^{k_i} \\ 1 & X_i \end{pmatrix}.$$

Let $P(\vec{X}) = (P_1(X_1), P_2(X_2), \dots, P_f(X_f))$, where indices are viewed mod f , and choose the type of the matrix $P_i(X_i)$ as follows:

Case (i). V is induced.

- (1) If $\ell_1 = 0$, $P_1 = t_2$;
- (2) If $\ell_1 = k_1 > 0$, $P_1 = t_1$.

For $i = 2, 3, \dots, f-1$ we choose the type of the matrix P_i as follows:

- (1) If $\ell_i = 0$, then:
 - If an even number of coordinates of $(P_1, P_2, \dots, P_{i-1})$ is of even type, $P_i = t_2$;
 - If an odd number of coordinates of $(P_1, P_2, \dots, P_{i-1})$ is of even type, $P_i = t_1$.
- (2) If $\ell_i = k_i > 0$, then:
 - If an even number of coordinates of $(P_1, P_2, \dots, P_{i-1})$ is of even type, $P_i = t_1$;
 - If an odd number of coordinates of $(P_1, P_2, \dots, P_{i-1})$ is of even type, $P_i = t_2$.

Finally, we choose the type of the matrix P_0 as follows:

- (1) If $\ell_0 = 0$, then:
 - If an even number of coordinates of $(P_1, P_2, \dots, P_{f-1})$ is of even type, $P_0 = t_4$;
 - If an odd number of coordinates of $(P_1, P_2, \dots, P_{f-1})$ is of even type, $P_0 = t_3$.
- (2) If $\ell_0 = k_0 > 0$, then:
 - If an even number of coordinates of $(P_1, P_2, \dots, P_{f-1})$ is of even type, $P_0 = t_2$;
 - If an odd number of coordinates of $(P_1, P_2, \dots, P_{f-1})$ is of even type, $P_0 = t_1$.

Case (ii). V is split reducible and non-ordinary.

The $(f-1)$ -tuple $(P_1, P_2, \dots, P_{f-1})$ is chosen as in Case (i) above. If $\eta = \eta_c$ is the unramified character which maps the geometric Frobenius element Frob_{K_f} of G_{K_f} to c , we replace the entry p^{k_0} in the definition of the matrix P_0 by cp^{k_0} . The type of the matrix P_0 is chosen as follows:

- (1) If $\ell_0 = 0$, then:
 - If an even number of coordinates of $(P_1, P_2, \dots, P_{f-1})$ is of even type, $P_0 = t_3$;
 - If an odd number of coordinates of $(P_1, P_2, \dots, P_{f-1})$ is of even type, $P_0 = t_4$.
- (2) If $\ell_0 = k_0 > 0$, then:
 - If an even number of coordinates of $(P_1, P_2, \dots, P_{f-1})$ is of even type, $P_0 = t_1$;
 - If an odd number of coordinates of $(P_1, P_2, \dots, P_{f-1})$ is of even type, $P_0 = t_2$.

Recall that $k \geq p$ and let

$$(1.1) \quad m := \begin{cases} \lfloor \frac{k-1}{p-1} \rfloor & \text{if } k_i \neq p \text{ for some } i, \\ 0 & \text{if } k_i = p \text{ for all } i. \end{cases}$$

For any $\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_f) \in (p^m \mathfrak{m}_E)^f$, let $P(\vec{\alpha})$ be the matrix obtained by evaluating each indeterminate X_i at α_i . In [8], we defined $\mathcal{F}(V)$ as the family of 2-dimensional crystalline representations $\{V(\vec{\alpha}), \vec{\alpha} \in (p^m \mathfrak{m}_E)^f\}$ corresponding by the Colmez-Fontaine theorem to the family of weakly admissible filtered φ -modules obtained by equipping $\mathbb{D}(\vec{\alpha}) = E^{|\tau|} \eta_1 \oplus E^{|\tau|} \eta_2$ with the Frobenius action defined by $(\varphi(\eta_1), \varphi(\eta_2)) = (\eta_1, \eta_2) P(\vec{\alpha})$ and the filtration

$$\text{Fil}^j(\mathbb{D}(\vec{\alpha})) = \begin{cases} E^{|\tau|} \eta_1 \oplus E^{|\tau|} \eta_2 & \text{if } j \leq 0, \\ E^{|\tau|s|}(\vec{x}\eta_1 + \vec{y}\eta_2) & \text{if } 1 + w_{s-1} \leq j \leq w_s, \text{ for } s = 0, 1, \dots, t-1, \\ 0 & \text{if } j \geq 1 + w_{t-1}, \end{cases}$$

where $\vec{x} = (x_0, x_1, \dots, x_{f-1})$ and $\vec{y} = (y_0, y_1, \dots, y_{f-1})$, with

$$(x_i, y_i) = \begin{cases} (1, -\alpha_i) & \text{if } P_i \text{ has type 1 or 2,} \\ (-\alpha_i, 1) & \text{if } P_i \text{ has type 3 or 4,} \end{cases}$$

for any $\vec{\alpha} \in (p^m \mathbf{m}_E)^f$. By the construction of these families in [8] it follows that $V(\vec{0}) = V$. We now show how to enlarge each such family $\mathcal{F}(V)$, leaving the modulo p reductions with respect to appropriately chosen Galois-stable \mathcal{O}_E -lattices unchanged, and preserving the Hodge-Tate types.

Let $\alpha(k) := \sum_{n=0}^{\infty} \lfloor \frac{k}{p^n(p-1)} \rfloor$. For any $A = (A_1, A, \dots, A_f) \in M_2(p^{\alpha(k-1)} \mathcal{O}_E)^{|\tau|}$ we define

$$P_A(\vec{X}) := (I\vec{d} + A) P(\vec{X}).$$

If $\vec{\alpha} \in (p^m \mathbf{m}_E)^f$ and $A \in M_2(p^{\alpha(k-1)} \mathcal{O}_E)^{|\tau|}$, we denote by $(\mathbb{D}_A(\vec{\alpha}), \varphi)$ the filtered φ -module obtained by equipping $\mathbb{D}_A(\vec{\alpha}) = E^{|\tau|} \eta_1 \oplus E^{|\tau|} \eta_2$ with the Frobenius endomorphism defined by $(\varphi(\eta_1), \varphi(\eta_2)) = (\eta_1, \eta_2) P_A(\vec{\alpha})$ and with the same filtration as $(\mathbb{D}(\vec{\alpha}), \varphi)$ independently of A . Such a filtered φ -module turns out to be weakly admissible. Let $V_A(\vec{\alpha})$ be the crystalline representation corresponding by the Colmez-Fontaine theorem to $(\mathbb{D}_A(\vec{\alpha}), \varphi)$, and let

$$\mathcal{G}(V) = \bigcup_{A \in \mathcal{M}(k)} \left\{ V_A(\vec{\alpha}) : \vec{\alpha} \in (p^m \mathbf{m}_E)^f \right\}, \text{ where } \mathcal{M}(k) = M_2(p^{1+\alpha(k-1)} \mathcal{O}_E)^{|\tau|}.$$

- Theorem A.** (i) For any $\vec{\alpha} \in (p^m \mathbf{m}_E)^f$ and any $A \in M_2(p^{\alpha(k-1)} \mathcal{O}_E)^{|\tau|}$ the filtered φ -modules $\mathbb{D}_A(\vec{\alpha})$ are weakly admissible and the corresponding crystalline representations have Hodge-Tate type $\text{HT}(\tau_i) = \{0, -k_i\}$;
- (ii) For any $\vec{\alpha} \in (p^m \mathbf{m}_E)^f$ and any $A \in M_2(p^{\alpha(k-1)} \mathcal{O}_E)^{|\tau|}$ there exist G_{K_f} -stable \mathcal{O}_E -lattices with respect to which $\overline{V}_A(\vec{\alpha}) = \overline{V}_A(\vec{0})$;
- (iii) There exist G_{K_f} -stable \mathcal{O}_E -lattices with respect to which all members of $\mathcal{G}(V)$ have the same modulo p reduction $\overline{\mathcal{G}}(\overline{V})$. Moreover, $\overline{\mathcal{G}}(\overline{V}) = \overline{V}$.

Remark 1.2. (1) By ([8, Theorems 1.5 & 1.7]),

$$\left(\overline{\mathcal{G}}(\overline{V})|_{I_{K_f}} \right)^{ss} = \begin{cases} \omega_{2f, \bar{\tau}_0}^\beta \oplus \omega_{2f, \bar{\tau}_0}^{p^f \beta}, & \text{where } \beta = -\sum_{i=0}^{2f-1} \ell_i p^i \text{ if } V \text{ is irreducible and induced,} \\ \omega_{f, \bar{\tau}_0}^\beta \oplus \omega_{f, \bar{\tau}_0}^{\beta'}, & \text{where } \beta = -\sum_{i=0}^{f-1} \ell_i p^i \text{ and } \beta' = -\sum_{i=0}^{f-1} \ell_{i+f} p^i, \text{ if } V \text{ is} \\ \text{split-reducible and nonordinary,} \end{cases}$$

where the level f fundamental character $\omega_{f, \bar{\tau}_0} : I_{K_f} \rightarrow k_E^\times$ is obtained by composing the homomorphism $I_{K_f} \rightarrow k_{K_f}^\times$ obtained from local class field theory (with uniformizers corresponding to geometric Frobenius elements) with the embedding of residue fields $k_{K_f} \xrightarrow{\bar{\tau}_0} k_E$ obtained by the distinguished embedding $K \xrightarrow{\tau_0} E$.

- (2) For the rest of this remark assume that $f \geq 2$. For a 2-dimensional crystalline $\overline{\mathbb{Q}_p}$ -representation V of G_{K_f} , the characteristic polynomial of Frobenius and a choice of the filtration of the corresponding by the Colmez-Fontaine theorem weakly admissible filtered φ -module $\mathbb{D}(V)$ fail to determine its isomorphism class. Assuming that $\mathbb{D}(V)$ is Frobenius-semisimple and non-Frobenius-scalar, and fixing the characteristic polynomial of Frobenius and a choice for the filtration, the additional datum required to determine the isomorphism class of V is (roughly) an element of $\mathbb{P}^{f-1}(E)$ (for a precise statement see [7, §7]). The isomorphism

classes of non-Frobenius-semisimple or Frobenius-scalar filtered φ -modules are in general messier to describe (see [7, §6]).

- (3) The representations of $\mathcal{G}(V)$ yield additional “projective parameters” compared to the set of “projective parameters” attached to the Frobenius-semisimple and non-Frobenius-scalar members of $\mathcal{F}(V)$. However, they yield no new characteristic polynomials or filtrations.
- (4) The formulas for the “projective parameters” of the Frobenius-semisimple and non-Frobenius-scalar representations of the families $\mathcal{G}(V)$ look particularly abhorrent (see for instance the proof of [8, Proposition 6.21]). The situation becomes even worse with the non-Frobenius-semisimple, and in especially with the Frobenius-scalar members of these families. This makes it hard to give a clean description, in terms of the classification of weakly admissible filtered φ -modules obtained in [7], of how many 2-dimensional crystalline representations of G_{K_f} with Hodge-Tate weights in the range $[0; p]$ we are able to compute the semisimplified modulo p reduction of, using Theorem A, and what is possibly missing.
- (5) Theorem A can be thought of as a local constancy result for the modulo p reductions of 2-dimensional crystalline representations of G_{K_f} within certain families. For results of similar flavor for 2-dimensional crystalline representations of $G_{\mathbb{Q}_p}$, see [2].

2. FAMILIES OF WACH MODULES

2.1. Étale (φ, Γ) -modules and Wach modules. Let $\mathcal{K}_n = K(\zeta_{p^n})$, where ζ_{p^n} is a primitive p^n -th root of unity inside $\overline{\mathbb{Q}_p}$ and let $K_\infty = \bigcup_{n \geq 1} \mathcal{K}_n$. Let $\chi : G_K \rightarrow \mathbb{Z}_p^\times$ be the cyclotomic character, $H_K := \ker \chi = \text{Gal}(\overline{\mathbb{Q}_p}/K_\infty)$ and $\Gamma_K := G_K/H_K = \text{Gal}(K_\infty/K)$. Fontaine [10] has constructed topological rings \mathbb{A} and \mathbb{B} endowed with continuous commuting Frobenius φ and $G_{\mathbb{Q}_p}$ -actions. Unless otherwise stated and whenever applicable, continuity means continuity with respect to the topologies induced by the weak topologies of the topological rings \mathbb{A} and \mathbb{B} . Let $\mathbb{A}_K = \mathbb{A}^{H_K}$ and $\mathbb{B}_K = \mathbb{B}^{H_K}$, and let $\mathbb{A}_{K,E} := \mathcal{O}_E \otimes_{\mathbb{Z}_p} \mathbb{A}_K$ and $\mathbb{B}_{K,E} := E \otimes_{\mathbb{Q}_p} \mathbb{B}_K$. The actions of φ and Γ_K extend to $\mathbb{A}_{K,E}$ and $\mathbb{B}_{K,E}$ by \mathcal{O}_E (resp. E)-linearity, and one easily sees that $\mathbb{A}_{K,E} = \mathbb{A}_E^{H_K}$ and $\mathbb{B}_{K,E} = \mathbb{B}_E^{H_K}$.

Definition 2.1. A (φ, Γ) -module over $\mathbb{A}_{K,E}$ (resp. $\mathbb{B}_{K,E}$) is an $\mathbb{A}_{K,E}$ -module of finite type (resp. a free $\mathbb{B}_{K,E}$ -module of finite type) endowed with a semilinear and continuous action of Γ_K , and with a semilinear map φ which commutes with the action of Γ_K . A (φ, Γ) -module M over $\mathbb{A}_{K,E}$ is called *étale* if $\varphi^*(M) = M$, where $\varphi^*(M)$ is the $\mathbb{A}_{K,E}$ -module generated by the set $\varphi(M)$. A (φ, Γ) -module M over $\mathbb{B}_{K,E}$ is called *étale* if it contains a basis (e_1, \dots, e_d) over $\mathbb{B}_{K,E}$ such that $(\varphi(e_1), \dots, \varphi(e_d)) = (e_1, \dots, e_d)A$ for some matrix $A \in \text{GL}_d(\mathbb{A}_{K,E})$.

If V is a continuous E -linear representation of G_K we equip the $\mathbb{B}_{K,E}$ -module $\mathbb{D}(V) := (\mathbb{B}_E \otimes_E V)^{H_K}$ with a Frobenius endomorphism φ defined by $\varphi(b \otimes v) := \varphi(b) \otimes v$, where φ on the right hand side is the Frobenius of \mathbb{B}_E , and with an action of Γ_K given by $\bar{g}(b \otimes v) := gb \otimes gv$ for any $g \in G_K$. This Γ_K -action commutes with φ and is continuous. Moreover, $\mathbb{D}(V)$ is an étale (φ, Γ) -module over $\mathbb{B}_{K,E}$. Conversely, if D is an étale (φ, Γ) -module over $\mathbb{B}_{K,E}$, let $\mathbb{V}(D) := (\mathbb{B}_E \otimes_{\mathbb{B}_{K,E}} D)^{\varphi=1}$, where $\varphi(b \otimes d) := \varphi(b) \otimes \varphi(d)$. The E -vector space $\mathbb{V}(D)$ is finite dimensional and is equipped with a continuous E -linear G_K -action given by $g(b \otimes d) := gb \otimes \bar{g}d$. We have the following theorem of Fontaine.

Theorem 2.2. [10]

- (i) *There is an equivalence of categories between continuous E -linear representations of G_K and étale (φ, Γ) -modules over $\mathbb{B}_{K,E}$ given by*

$$\mathbb{D} : \text{Rep}_E(G_K) \rightarrow \text{Mod}_{(\varphi, \Gamma)}^{\text{ét}}(\mathbb{B}_{K,E}) : V \mapsto \mathbb{D}(V) := (\mathbb{B}_E \otimes_E V)^{H_K},$$

with quasi-inverse functor

$$\mathbb{V} : \text{Mod}_{(\varphi, \Gamma)}^{\text{ét}}(\mathbb{B}_{K,E}) \rightarrow \text{Rep}_E(G_K) : D \mapsto \mathbb{V}(D) := (\mathbb{B}_E \otimes_{\mathbb{B}_{K,E}} D)^{\varphi=1}.$$

- (ii) *There is an equivalence of categories between continuous \mathcal{O}_E -linear representations of G_K and étale (φ, Γ) -modules over $\mathbb{A}_{K,E}$ given by*

$$\mathbb{D} : \text{Rep}_{\mathcal{O}_E}(G_K) \rightarrow \text{Mod}_{(\varphi, \Gamma)}^{\text{ét}}(\mathbb{A}_{K,E}) : T \mapsto \mathbb{D}(T) := (\mathbb{A}_E \otimes_{\mathcal{O}_E} T)^{H_K},$$

with quasi-inverse functor

$$\mathbb{T} : \text{Mod}_{(\varphi, \Gamma)}^{\text{ét}}(\mathbb{A}_{K,E}) \rightarrow \text{Rep}_{\mathcal{O}_E}(G_K) : D \mapsto \mathbb{T}(D) := (\mathbb{A}_E \otimes_{\mathbb{A}_{K,E}} D)^{\varphi=1}.$$

Let $\mathbb{A}_K = \{\sum_{n=-\infty}^{+\infty} \alpha_n \pi^n : \alpha_n \in \mathcal{O}_K \text{ and } \lim_{n \rightarrow -\infty} \alpha_n = 0\}$ for some element π which can be thought of as a formal variable. The ring \mathbb{A}_K is equipped with a Frobenius endomorphism φ which extends the absolute Frobenius of \mathcal{O}_K and is such that $\varphi(\pi) = (1 + \pi)^p - 1$. It is also equipped with a Γ_K -action which is \mathcal{O}_K -linear, commutes with Frobenius, and is such that $\gamma(\pi) = (1 + \pi)^{\chi(\gamma)} - 1$ for all $\gamma \in \Gamma_K$. The ring \mathbb{A}_K is a local domain with maximal ideal (p) and fraction field $\mathbb{B}_K = \mathbb{A}_K[\frac{1}{p}]$. The rings \mathbb{A}_K , $\mathbb{A}_{K,E}$, \mathbb{B}_K and $\mathbb{B}_{K,E}$ contain the subrings $\mathbb{A}_K^+ = \mathcal{O}_K[[\pi]]$, $\mathbb{A}_{K,E}^+ := \mathcal{O}_E \otimes_{\mathbb{Z}_p} \mathbb{A}_K^+$, $\mathbb{B}_K^+ = \mathbb{A}_K^+[\frac{1}{p}]$ and $\mathbb{B}_{K,E}^+ := E \otimes_{\mathbb{Q}_p} \mathbb{B}_K^+$ respectively, and these subrings are equipped with the restrictions of the φ and the Γ_K -actions of the rings containing them. The map $v : \mathbb{A}_{K,E}^+ \rightarrow \prod_{\tau: K \hookrightarrow E} \mathcal{O}_E[[\pi]]$ given by $v(a \otimes b) = (a\tau_0(b), a\tau_1(b), \dots, a\tau_{f-1}(b))$, where $\tau_i(\sum_{n=0}^{\infty} \beta_n \pi^n) = \sum_{n=0}^{\infty} \tau_i(\beta_n) \pi^n$ for all $b = \sum_{n=0}^{\infty} \beta_n \pi^n \in \mathbb{A}_K^+$ is a ring isomorphism. The ring $\mathcal{O}_E[[\pi]]^{|\tau|} := \prod_{\tau: K \hookrightarrow E} \mathcal{O}_E[[\pi]]$ is equipped via v with commuting \mathcal{O}_E -linear actions of φ and Γ_K given by the formulas

$$(2.1) \quad \varphi(\alpha_0(\pi), \alpha_1(\pi), \dots, \alpha_{f-1}(\pi)) = (\alpha_1(\varphi(\pi)), \dots, \alpha_{f-1}(\varphi(\pi)), \alpha_0(\varphi(\pi))) \text{ and}$$

$$(2.2) \quad \gamma(\alpha_0(\pi), \alpha_1(\pi), \dots, \alpha_{f-1}(\pi)) = (\alpha_0(\gamma\pi), \alpha_1(\gamma\pi), \dots, \alpha_{f-1}(\gamma\pi))$$

for all $\gamma \in \Gamma_K$.

Definition 2.3. *Suppose $k \geq 0$. A Wach module over $\mathbb{A}_{K,E}^+$ (resp. $\mathbb{B}_{K,E}^+$) with weights in $[-k; 0]$ is a free $\mathbb{A}_{K,E}^+$ -module (resp. $\mathbb{B}_{K,E}^+$ -module) N of finite rank, endowed with an action of Γ_K which becomes trivial modulo π , and also with a Frobenius map φ which commutes with the action of Γ_K and such that $\varphi(N) \subset N$ and $N/\varphi^*(N)$ is killed by q^k , where $q := \varphi(\pi)/\pi$ and $\varphi^*(N)$ is the $\mathbb{A}_{K,E}^+$ -module (resp. $\mathbb{B}_{K,E}^+$ -module) generated by the set $\varphi(N)$.*

The following theorem of Berger determines which types of étale (φ, Γ) -modules correspond to crystalline representations via Fontaine's functor.

Theorem 2.4. [1]

- (i) *An E -linear representation V of G_K is crystalline with Hodge-Tate weights in $[-k; 0]$ if and only if $\mathbb{D}(V)$ contains a unique Wach module $\mathbb{N}(V)$ of rank $\dim_E V$ with weights in $[-k; 0]$. The functor $V \mapsto \mathbb{N}(V)$ defines an equivalence of categories between crystalline representations of G_K and Wach modules over $\mathbb{B}_{K,E}^+$, compatible with tensor products, duality and exact sequences.*

- (ii) For a given crystalline E -representation V , the map $T \mapsto \mathbb{N}(T) := \mathbb{N}(V) \cap \mathbb{D}(T)$ induces a bijection between G_K -stable, \mathcal{O}_E -lattices of V and Wach modules over $\mathbb{A}_{K,E}^+$ which are $\mathbb{A}_{K,E}^+$ -lattices contained in $\mathbb{N}(V)$. Moreover $\mathbb{D}(T) = \mathbb{A}_{K,E} \otimes_{\mathbb{A}_{K,E}^+} \mathbb{N}(T)$.
- (iii) If V is a crystalline E -representation of G_K , and if we endow $\mathbb{N}(V)$ with the filtration $\text{Fil}^j \mathbb{N}(V) = \{x \in \mathbb{N}(V) | \varphi(x) \in q^j \mathbb{N}(V)\}$, then we have an isomorphism

$$\mathbb{D}_{\text{cris}}(V) \rightarrow \mathbb{N}(V)/\pi \mathbb{N}(V)$$

of filtered φ -modules over $E^{|\tau|}$ (with the induced filtration on $\mathbb{N}(V)/\pi \mathbb{N}(V)$).

2.2. Construction of families of Wach modules. We fix a topological generator δ of the pro-cyclic group Γ_K . For any positive integer ℓ , let $\alpha(\ell) := \sum_{j=1}^{\ell} v_p(1 - \chi(\delta))^j$ and let $\alpha(0) = 0$. Recall that $\alpha(\ell) = 0$ for $\ell \leq p-2$, while for an arbitrary ℓ , $\alpha(\ell) = \sum_{n=0}^{\infty} \lfloor \frac{\ell}{p^n(p-1)} \rfloor \leq \lfloor \frac{\ell p}{(p-1)^2} \rfloor$ (cf. [1, §IV.1]). Let \mathcal{S} be a set of indeterminates and let π be a distinguished indeterminate not belonging to \mathcal{S} . We denote by $M_n^{\mathcal{S}}$ the matrix ring $M_n(\mathcal{O}_E[[\pi, \mathcal{S}]])^{|\tau|}$. Recall that $k := \max\{k_i\} \geq p$. For any integer $s \geq 0$ we write $\vec{\pi}^s = (\pi^s, \pi^s, \dots, \pi^s)$, and we denote by \vec{Id} the matrix of $M_n^{\mathcal{S}}$ whose coordinates are the identity matrix. We need the following variant of ([3, Lemma 10.3.2]).

Lemma 2.5. *For each $\gamma \in \Gamma_K$, let $G_\gamma = G_\gamma(\mathcal{S}) \in \vec{Id} + \vec{\pi} M_n^{\mathcal{S}}$. Let $c \geq 0$ be an integer and let $A = (A_1, A_2, \dots, A_f)$ be any matrix in $M_n(p^{c+\alpha(k-1)} \mathcal{O}_E)^{|\tau|}$. There exists a matrix*

$$\hat{A} = (\hat{A}_1, \hat{A}_2, \dots, \hat{A}_f) \in M_n(p^c \mathcal{O}_E[[\pi, \mathcal{S}]])^{|\tau|}$$

such that:

- (i) $\hat{A} \equiv A \pmod{\vec{\pi}}$;
- (ii) $\vec{Id} + \hat{A} \in \text{GL}_n(\mathcal{O}_E[[\pi, \mathcal{S}]])^{|\tau|}$;
- (iii) $(\vec{Id} + \hat{A}) \cdot G_\gamma \cdot \gamma(\vec{Id} + \hat{A})^{-1} \equiv G_\gamma \pmod{\vec{\pi}^k}$;
- (iv) If $A \in M_n(p^{1+\alpha(k-1)} \mathcal{O}_E)^{|\tau|}$, then $\hat{A} \equiv 0 \pmod{p}$ and

$$G_\gamma - (\vec{Id} + \hat{A}) \cdot G_\gamma \cdot \gamma(\vec{Id} + \hat{A})^{-1} \equiv 0 \pmod{p}.$$

Proof. Since $k \geq p$, for any $A \in M_n(p^{c+\alpha(k-1)} \mathcal{O}_E)^{|\tau|}$ it follows that $\vec{Id} + A \in \text{GL}_n(\mathcal{O}_E)^{|\tau|}$. Let $\hat{A}_i = A_i + \pi \hat{A}_i^1 + \pi^2 \hat{A}_i^2 + \dots + \pi^{k-1} \hat{A}_i^{k-1}$ and $G_\gamma = (G_\gamma^1, G_\gamma^2, \dots, G_\gamma^f)$, with $G_\gamma^i = Id + \pi G_1^i + \pi^2 G_2^i + \dots + \pi^{k-1} G_{k-1}^i + \dots$ (suppressing the dependence of the G_j^i on γ). We first show that $\vec{Id} + \hat{A} \in \text{GL}_n(\mathcal{O}_E[[\pi, \mathcal{S}]])^{|\tau|}$. Since $Id + A_i \in \text{GL}_n(p^{c+\alpha(k-1)} \mathcal{O}_E)$ for all i , each coordinate matrix $Id + \hat{A}_i$ is invertible and the inverse is $(Id + \hat{A}_i)^{-1} = (\sum_{n=0}^{\infty} (-1)^n \pi^n B_i^n) (Id + A_i)^{-1}$, where

$$B_i = (Id + A_i)^{-1} (\hat{A}_i^1 + \pi \hat{A}_i^2 + \dots + \pi^{k-2} \hat{A}_i^{k-1}).$$

To prove part (iii), we need to choose the matrices \hat{A}_i^j so that

$$\begin{aligned} & (Id + \pi G_1^i + \pi^2 G_2^i + \dots + \pi^{k-1} G_{k-1}^i + \dots) \gamma (A_i + \pi \hat{A}_i^1 + \pi^2 \hat{A}_i^2 + \dots + \pi^{k-1} \hat{A}_i^{k-1}) = \\ & (A_i + \pi \hat{A}_i^1 + \pi^2 \hat{A}_i^2 + \dots + \pi^{k-1} \hat{A}_i^{k-1}) (Id + \pi G_1^i + \pi^2 G_2^i + \dots + \pi^{k-1} G_{k-1}^i + \dots) \pmod{\pi^k}. \end{aligned}$$

We may assume that γ is a topological generator of Γ_K . We solve for the \hat{A}_i^j , bearing in mind that $\gamma(\pi)^r \equiv \chi(\gamma)^r \pi^r \pmod{\pi^{r+1}}$ for all $r \geq 1$. First, we solve for $\hat{A}_i^1 \in M_n(\mathcal{O}_E[[\mathcal{S}]])$ so that $(1 - \chi(\gamma)) \hat{A}_i^1 = A_i G_1^i - G_1^i A_i$. Since $A_i G_1^i - G_1^i A_i \in M_n(p^{c+\alpha(k-1)} \mathcal{O}_E[[\mathcal{S}]])$, we see that $\hat{A}_i^1 := (1 - \chi(\gamma))^{-1} (A_i G_1^i - G_1^i A_i)$ and $\hat{A}_i^1 \in M_n(p^{c+\alpha(k-1)-v_p(1-\chi(\gamma))} \mathcal{O}_E[[\mathcal{S}]])$. We then solve for \hat{A}_i^2 so that $\left((1 - \chi(\gamma)^2)\right) \hat{A}_i^2$ is an \mathcal{O}_E -linear combination of products of $A_i, \hat{A}_i^1, G_1^i, G_2^i$ which belong to $M_n(p^{c+\alpha(k-1)-v_p(1-\chi(\gamma))} \mathcal{O}_E[[\mathcal{S}]])$. Dividing this linear combination by $(1 - \chi(\gamma)^2)$, we get

$$\hat{A}_i^2 \in M_n\left(p^{c+\alpha(k-1)-v_p((1-\chi(\gamma))(1-\chi(\gamma)^2))} \mathcal{O}_E[[\mathcal{S}]])\right).$$

Continuing this way we solve for $\hat{A}_i^{k-1} \in M_n(p^c \mathcal{O}_E[[\mathcal{S}]])$. Part (iv) is clear. \square

Let $c_i \in \mathcal{O}_E^\times$ and let $\Pi(\mathcal{S}) = (\Pi_1, \Pi_2, \dots, \Pi_f) \in M_n^{\mathcal{S}}$ with $\det(\Pi_i) = c_i q^{k_i}$, where $q = \frac{(1+\pi)^p - 1}{\pi}$. We denote by I the ideal of $M_n(\mathcal{O}_E[[\mathcal{S}]])$ generated by the set $\{p \cdot Id, X_i \cdot Id : X_i \in \mathcal{S}\}$ and by \overline{M}_n the quotient ring of $M_n(\mathcal{O}_E[[\mathcal{S}]])$ modulo I . Letting φ act trivially on the elements of \mathcal{S} , and letting $\varphi(\pi) = (1 + \pi)^p - 1$, we have

$$(2.3) \quad \varphi(\vec{\alpha}) = (\varphi(\alpha_1), \varphi(\alpha_2), \dots, \varphi(\alpha_0))$$

for all $\vec{\alpha} = (\alpha_0, \alpha_1, \dots, \alpha_{f-1}) \in \mathcal{O}_E[[\pi, \mathcal{S}]]^{|\tau|}$. We denote $\text{Nm}_\varphi(\vec{\alpha}) := \vec{\alpha} \cdot \varphi(\vec{\alpha}) \cdots \varphi^{f-1}(\vec{\alpha})$ and for any matrix A in $M_n^{\mathcal{S}}$ we write $\text{Nm}_\varphi(A) := A \cdot \varphi(A) \cdots \varphi^{f-1}(A)$, with φ acting on each entry of the matrix A as in formula (2.3). We fix a matrix $\Pi(\mathcal{S}) \in M_n^{\mathcal{S}}$ as above. For the rest of this section we assume that for any $\gamma \in \Gamma_K$ there exists a matrix $G_\gamma^{(k)} = G_\gamma^{(k)}(\mathcal{S}) \in M_n^{\mathcal{S}}$ such that:

- (a) $G_\gamma^{(k)}(\mathcal{S}) \equiv \vec{Id} \pmod{\vec{\pi}}$;
- (b) $G_\gamma^{(k)}(\mathcal{S}) - \Pi(\mathcal{S}) \varphi(G_\gamma^{(k)}(\mathcal{S})) \gamma(\Pi(\mathcal{S})^{-1}) \in \vec{\pi}^k M_n^{\mathcal{S}}$;
- (c) There exist no nonzero matrix $B \in M_n(\mathcal{O}_E[[\mathcal{S}]])^{|\tau|}$ and integer $t > 0$ such that $BU = p^{ft} UB$, where $U = \text{Nm}_\varphi(\Pi(\mathcal{S}))$;
- (d) If $k = k_i$ for all i , we additionally assume that the operator

$$(2.4) \quad \overline{H} \mapsto \overline{H - Q_f H (p^{fk} Q_f^{-1})} : \overline{M}_n \rightarrow \overline{M}_n$$

is surjective. Let $R_\gamma^{(k)}(\mathcal{S})$ be the matrices defined by

$$(2.5) \quad \vec{\pi}^k R_\gamma^{(k)}(\mathcal{S}) := G_\gamma^{(k)}(\mathcal{S}) - \Pi(\mathcal{S}) \varphi(G_\gamma^{(k)}(\mathcal{S})) \gamma(\Pi(\mathcal{S})^{-1}).$$

Proposition 2.6. *Let $A = (A_1, A_2, \dots, A_f) \in M_n(p^{\alpha(k-1)} \mathcal{O}_E)^{|\tau|}$ and let $Q_f^A := \prod_{i=1}^f (Id + A_i) P_i$, where $P_i = \Pi_i \pmod{\pi}$ for all i . Assume that*

- (1) *There exist no nonzero matrix $B \in M_n(\mathcal{O}_E[[\mathcal{S}]])^{|\tau|}$ and integer $t > 0$ such that $BU_A = p^{ft} U_A B$, where $U_A = \text{Nm}_\varphi\left((Id + A) P(\mathcal{S})\right)$.*

If $n = 2$, we replace assumption (1) by the following assumption.

- (1) $\text{Tr}(Q_f^A) \notin \overline{\mathbb{Q}_p}$.

Let $\hat{A} \in M_n(\mathcal{O}_E[[\pi, \mathcal{S}]])^{|\tau|}$ be as in Lemma 2.5 applied for the matrices $G_\gamma(\mathcal{S}) := G_\gamma^{(k)}(\mathcal{S})$, where the $G_\gamma^{(k)}(\mathcal{S})$ are as in the assumptions preceding this proposition, and let $\Pi_{\hat{A}}(\mathcal{S}) := (Id + \hat{A}) \Pi(\mathcal{S})$. Then for each $\gamma \in \Gamma_K$ there exists a unique matrix $G_{\gamma, \hat{A}}(\mathcal{S}) \in M_n^{\mathcal{S}}$ such that

- (i) $G_{\gamma, \hat{A}}(\mathcal{S}) \equiv \overrightarrow{Id} \bmod \bar{\pi}$ and
- (ii) $\Pi_{\hat{A}}(\mathcal{S})\varphi(G_{\gamma, \hat{A}}(\mathcal{S})) = G_{\gamma, \hat{A}}(\mathcal{S})\gamma(\Pi_{\hat{A}}(\mathcal{S}))$.

Proof. Since $k \geq p$, it follows that $\det(Id + A_i) \in \mathcal{O}_E^\times$ for all i . Let $G_{\gamma, \hat{A}}^{(k)}(\mathcal{S}) := G_{\gamma}^{(k)}(\mathcal{S})$ for all $\gamma \in \Gamma_K$. The proposition follows from [8, Lemma 4.4] applied for the matrices $\Pi_{\hat{A}}(\mathcal{S})$ and $G_{\gamma, \hat{A}}^{(k)}(\mathcal{S})$ for the case where $\ell = k$. Assumption (1) of this lemma clearly holds. If $k = k_i$ for all i , since $k \geq p$, the operator

$$(2.6) \quad \overline{H} \mapsto \overline{H - Q_f^A H (p^{fk} (Q_f^A)^{-1})} : \overline{M_n} \rightarrow \overline{M_n}$$

coincides with the operator (2.4) which was assumed to be surjective. Hence assumption (4) of [8, Lemma 4.4] holds. If $n = 2$, assumption (3) holds because of assumption (1) and [8, Corollary 5.3]. Lemma 2.5 (ii) implies that $Id + \hat{A} \in \mathrm{GL}_n(\mathcal{O}_E[[\pi, \mathcal{S}]])^{|\tau|}$. Moreover,

$$\begin{aligned} G_{\gamma, \hat{A}}^{(k)}(\mathcal{S}) - \Pi_{\hat{A}}(\mathcal{S}) \cdot \varphi\left(G_{\gamma, \hat{A}}^{(k)}(\mathcal{S})\right) \cdot \gamma\left(\Pi_{\hat{A}}(\mathcal{S})^{-1}\right) &= \\ G_{\gamma}^{(k)}(\mathcal{S}) - \left(Id + \hat{A}\right) \cdot \Pi(\mathcal{S}) \cdot \varphi\left(G_{\gamma}^{(k)}(\mathcal{S})\right) \cdot \gamma \Pi(\mathcal{S})^{-1} \cdot \gamma\left(Id + \hat{A}\right)^{-1} &\stackrel{(2.5)}{=} \\ G_{\gamma}^{(k)}(\mathcal{S}) - \left(Id + \hat{A}\right) \cdot G_{\gamma}^{(k)}(\mathcal{S}) \cdot \gamma\left(Id + \hat{A}\right)^{-1} + \bar{\pi}^k \cdot \left(Id + \hat{A}\right) \cdot R_{\gamma}^{(k)}(\mathcal{S}) \cdot \gamma\left(Id + \hat{A}\right)^{-1}. \end{aligned}$$

By Lemma 2.5 (iii), $G_{\gamma}^{(k)}(\mathcal{S}) - \left(Id + \hat{A}\right) \cdot G_{\gamma}^{(k)}(\mathcal{S}) \cdot \gamma\left(Id + \hat{A}\right)^{-1} \in \bar{\pi}^k M_n^{\mathcal{S}}$ and therefore assumption (2) of [8, Lemma 4.4] holds. This completes the proof. \square

Proposition 2.7. *For any $\vec{a} = (a_0, a_1, \dots, a_{f-1}) \in \mathfrak{m}_E^{|\mathcal{S}|}$ and any $\gamma_1, \gamma_2, \gamma \in \Gamma_K$, the following equations hold:*

- (i) $G_{\gamma_1 \gamma_2, \hat{A}}(\vec{a}) = G_{\gamma_1, \hat{A}}(\vec{a}) \gamma_1(G_{\gamma_2, \hat{A}}(\vec{a}))$ and
- (ii) $\Pi_{\hat{A}}(\vec{a})\varphi(G_{\gamma, \hat{A}}(\vec{a})) = G_{\gamma, \hat{A}}(\vec{a})\gamma(\Pi_{\hat{A}}(\vec{a}))$.

Proof. Both matrices $G_{\gamma_1 \gamma_2, \hat{A}}(\mathcal{S})$ and $G_{\gamma_1, \hat{A}}(\mathcal{S})\gamma_1(G_{\gamma_2, \hat{A}}(\mathcal{S}))$ are $\equiv \overrightarrow{Id} \bmod \bar{\pi}$ and are solutions in B of the equation $\Pi(\mathcal{S})\varphi(B) = B\gamma(\Pi(\mathcal{S}))$. They are equal by the uniqueness part of Proposition 2.6. The second equation follows from conclusion (ii) of the same proposition. \square

For any $\vec{a} \in \mathfrak{m}_E^{|\mathcal{S}|}$ we equip $\mathbb{N}_{\hat{A}}(\vec{a}) = \bigoplus_{i=1}^n (\mathcal{O}_E[[\pi]]^{|\tau|}) \eta_i$ with φ and Γ_K -actions defined by $(\varphi(\eta_1), \varphi(\eta_2), \dots, \varphi(\eta_n)) = (\eta_1, \eta_2, \dots, \eta_n) \Pi_{\hat{A}}(\vec{a})$ and $(\gamma\eta_1, \gamma\eta_2, \dots, \gamma\eta_n) = (\eta_1, \eta_2, \dots, \eta_n) G_{\gamma, \hat{A}}(\vec{a})$ respectively. Proposition 2.7 implies that $(\gamma_1 \gamma_2)x = \gamma_1(\gamma_2 x)$ and $\varphi(\gamma x) = \gamma(\varphi(x))$ for all $x \in \mathbb{N}_{\hat{A}}(\vec{a})$ and $\gamma, \gamma_1, \gamma_2 \in \Gamma_K$. Since $G_{\gamma, \hat{A}}(\vec{a}) \equiv \overrightarrow{Id} \bmod \bar{\pi}$, it follows that the Γ_K action on $\mathbb{N}_{\hat{A}}(\vec{a})$ is trivial modulo $\pi \mathbb{N}_{\hat{A}}(\vec{a})$. We have the following.

Proposition 2.8. *For any $\vec{a} \in \mathfrak{m}_E^{|\mathcal{S}|}$ the module $\mathbb{N}_{\hat{A}}(\vec{a})$ equipped with the φ and Γ_K -actions defined by $\Pi_{\hat{A}}(\vec{a})$ and $G_{\gamma, \hat{A}}(\vec{a})$ respectively is a Wach module corresponding to some G_K -stable \mathcal{O}_E -lattice inside some n -dimensional crystalline E -representation of G_K with Hodge-Tate weights in $[-k; 0]$.*

Proof. By Theorem 2.4, the only thing left to prove is that $q^k \mathbb{N}_{\hat{A}}(\vec{a}) \subset \varphi^*(\mathbb{N}_{\hat{A}}(\vec{a}))$. This is identical to the proof of [8, Proposition 4.6]. \square

Let $V_{\hat{A}}(\vec{a}) = E \otimes_{\mathcal{O}_E} T_{\hat{A}}(\vec{a})$, where $T_{\hat{A}}(\vec{a}) = \mathbb{T}(\mathbb{D}_{\hat{A}}(\vec{a}))$ and $\mathbb{D}_{\hat{A}}(\vec{a}) = \mathbb{A}_{K, E} \otimes_{\mathbb{A}_{K, E}^+} \mathbb{N}_{\hat{A}}(\vec{a})$. By Theorem 2.4 the representations $V_{\hat{A}}(\vec{a})$ are n -dimensional crystalline E -representations of G_K with

Hodge-Tate weights in $[-k; 0]$. They are independent of the lifting \hat{A} of A (see Remark 3.8 (i) below) and instead we denote them by $V_A(\vec{a})$. Regarding the mod p reductions of these representations, we have the following.

Proposition 2.9. *For any $\vec{a} \in \mathfrak{m}_E^{|\mathcal{S}|}$ there exist G_K -stable \mathcal{O}_E -lattices such that $\overline{V}_A(\vec{a}) \simeq \overline{V}_A(\vec{0})$.*

Proof. Identical to the proof of [8, Theorem 4.7], given that $\Pi_{\hat{A}}(\vec{a}) \equiv \Pi_{\hat{A}}(\vec{0}) \bmod \mathfrak{m}_E$ and $G_{\gamma, \hat{A}}(\vec{a}) \equiv G_{\gamma, \hat{A}}(\vec{0}) \bmod \mathfrak{m}_E$. \square

Lemma 2.10. *Let $A \in M_n(p^{1+\alpha(k-1)}\mathcal{O}_E)^{|\tau|}$. For any $\gamma \in \Gamma_K$ and for $\dagger \in \{\emptyset, \hat{A}\}$, let*

$$(2.7) \quad G_{\gamma, \dagger}^{(k)}(\mathcal{S}) - \Pi_{\dagger}(\mathcal{S}) \varphi \left(G_{\gamma, \dagger}^{(k)}(\mathcal{S}) \right) \gamma \left(\Pi_{\dagger}(\mathcal{S})^{-1} \right) =: \vec{\pi}^k R_{\gamma, \dagger}^{(k)}, \text{ where } G_{\gamma, \hat{A}}^{(k)}(\mathcal{S}) := G_{\gamma}^{(k)}(\mathcal{S}) \quad \forall \gamma.$$

Then (i) $\Pi_{\hat{A}}(\mathcal{S}) \equiv \Pi(\mathcal{S}) \bmod I$, and (ii) $R_{\gamma, \hat{A}}^{(k)}(\mathcal{S}) \equiv R_{\gamma}^{(k)}(\mathcal{S}) \bmod I$.

Proof. By Lemma 2.5 (iv), $\hat{A} \equiv 0 \bmod p$ and part (i) is clear. The lemma follows immediately reducing equations (2.7) mod I . \square

3. PROOF OF THE THEOREM

Throughout this section we assume that $n = 2$. For the rest of the paper we fix an f -tuple $P = (P_1, P_2, \dots, P_f)$ chosen as in §1. For any $A \in M_2(\mathcal{O}_E[[\pi]])^{|\tau|}$ we define $Q_f^A := \prod_{i=1}^{i=f} (Id + A_i) P_i$ and we let $Q_f := Q_f^0$. We define

$$m_k := \begin{cases} 0 & \text{if } k_j = p \text{ for all } j \text{ and } \text{Tr}(Q_f) \notin \overline{\mathbb{Q}_p}, \\ \lfloor \frac{k-1}{p-1} \rfloor & \text{otherwise.} \end{cases}$$

For matrices P chosen as in §1 the condition $\text{Tr}(Q_f) \notin \overline{\mathbb{Q}_p}$ in the definition of m_k turns out to be redundant (see Lemma 3.1 (i) below), and m_k coincides with the integer m defined in formula (1.1). Let $\Pi(\mathcal{S}) = (\Pi_1(S_1), \Pi_2(S_2), \dots, \Pi_f(S_f)) \in M_n^{\mathcal{S}}$, where Π_i are matrices of one of the following four types:

$$t_1: \begin{pmatrix} c_i q^{k_i} & 0 \\ S_i \varphi(z_i) & 1 \end{pmatrix}, \quad t_2: \begin{pmatrix} S_i \varphi(z_i) & 1 \\ c_i q^{k_i} & 0 \end{pmatrix}, \quad t_3: \begin{pmatrix} 1 & S_i \varphi(z_i) \\ 0 & c_i q^{k_i} \end{pmatrix}, \quad t_4: \begin{pmatrix} 0 & c_i q^{k_i} \\ 1 & S_i \varphi(z_i) \end{pmatrix},$$

with $S_i \in \mathcal{S}$, $c_i \in \mathcal{O}_E^\times$. The z_i are polynomials in $\mathbb{Z}_p[\pi]$ of degree $\leq k-1$ such that $z_i \equiv p^{m_k} \bmod \pi$, suitably chosen so that there exist matrices $G_{\gamma}^{(k)}(\mathcal{S}) \in M_n^{\mathcal{S}}$ with $G_{\gamma}^{(k)}(\mathcal{S}) \equiv Id \bmod \vec{\pi}$ such that $G_{\gamma}^{(k)}(\mathcal{S}) - \Pi(\mathcal{S}) \varphi(G_{\gamma}^{(k)}(\mathcal{S})) \gamma (\Pi(\mathcal{S})^{-1}) \in \vec{\pi}^k M_n^{\mathcal{S}}$. The existence of such polynomials has been established in [8, Proposition 5.9 & Remark 5.12]. We let $\vec{X} = (X_1, X_2, \dots, X_f)$ with $\vec{X} = p^{m_k} \vec{S}$, and we choose Π so that its modulo $\vec{\pi}$ reduction equals P . In particular, the type of Π_i coincides with the type of P_i for all i . If $A \in M_2(\mathcal{O}_E[[\pi]])^{|\tau|}$, let \hat{A} be a fixed choice of a lifting of A as in Lemma 2.5 with respect to a fixed choice of matrices $G_{\gamma}(\mathcal{S}) := G_{\gamma}^{(k)}(\mathcal{S})$ as above, and let $\Pi_{\hat{A}}(\mathcal{S}) = (Id + \hat{A}) \Pi(\mathcal{S})$. Let E_{ij} , $i, j = 1, 2$, be the 2×2 matrix with (i, j) entry 1 and all other entries 0.

Lemma 3.1. (i) $\text{Tr}(Q_f) \notin \overline{\mathbb{Q}_p}$;
(ii) $\text{Tr}(Q_f) \notin p\overline{\mathbb{Z}_p}[X_1, X_2, \dots, X_f]$.
(iii) For any $A \in M_2(p\mathcal{O}_E[[\pi]])^{|\tau|}$, $\text{Tr}(Q_f^A) \notin \overline{\mathbb{Q}_p}$;
(iv) For any $A \in M_2(p\mathcal{O}_E[[\pi]])^{|\tau|}$ the operator (2.6) is surjective.

Proof. For part (i) recall that in the proofs of [8, Theorems 1.5 & 1.7], the types of the coordinate matrices P_i of P have been chosen so that $\text{Tr}(Q_f) \notin \overline{\mathbb{Q}_p}$. For part (ii), we have

$$(3.1) \quad P_i \bmod p = \begin{cases} c(k_i) E_{11} + E_{22} + X_i E_{21} & \text{if } P_i = t_1, \\ c(k_i) E_{21} + E_{12} + X_i E_{11} & \text{if } P_i = t_2, \\ c(k_i) E_{22} + E_{11} + X_i E_{12} & \text{if } P_i = t_3, \\ c(k_i) E_{12} + E_{21} + X_i E_{22} & \text{if } P_i = t_4, \end{cases} \quad \text{where } c(k_i) = \begin{cases} 0 & \text{if } k_i > 0, \\ 1 & \text{if } k_i = 0. \end{cases}$$

The (i, i) entries in $Q_f \bmod p$ are sums of distinct terms of the form 1 and $X_{i_1} \cdot X_{i_2} \cdots X_{i_{r_i}}$ for some $1 \leq r_i \leq f$. Hence $\text{Tr}(Q_f) \not\equiv 0 \bmod p$ (if the diagonal entries of $\text{Tr}(Q_f) \bmod p$ coincide, we use that $p \neq 2$). For part (iii), assume that $\text{Tr}(Q_f^A) \in \overline{\mathbb{Q}_p}$. Since the entries of Q_f^A are in $\overline{\mathbb{Z}_p}[X_1, X_2, \dots, X_f]$ it follows that $\text{Tr}(Q_f^A) \in \overline{\mathbb{Z}_p}$. Since $Q_f^A \equiv Q_f \bmod p$ it follows that $\text{Tr}(Q_f^A) \equiv \text{Tr}(Q_f) \bmod p$ and therefore that $\text{Tr}(Q_f) \in \overline{\mathbb{Z}_p} + p\overline{\mathbb{Z}_p}[X_1, X_2, \dots, X_f]$. Since $\text{Tr}(Q_f) \notin \overline{\mathbb{Q}_p}$, [8, Lemma 5.19 & Corollary 5.17] imply that $Q_f \bmod I = E_{12}$ or E_{21} , therefore $\text{Tr}(Q_f) \equiv 0 \bmod (p, X_1, \dots, X_f)$. Hence $\text{Tr}(Q_f) \in (\overline{\mathbb{Z}_p} + p\overline{\mathbb{Z}_p}[X_1, X_2, \dots, X_f]) \cap (p, X_1, \dots, X_f) = p\overline{\mathbb{Z}_p}[X_1, X_2, \dots, X_f]$ which contradicts part (ii) of the lemma. Part (iv) for $A = 0$ follows from [8, Corollary 5.20]. The general case holds because the operators with any $A \in M_2(p\mathcal{O}_E[[\pi]])^{|\tau|}$ coincide with that with $A = 0$. \square

Proposition 3.2. *Let $A \in M_2(p^{\alpha(k-1)}\mathcal{O}_E[[\pi]])^{|\tau|}$ and let $\Pi_{\hat{A}}(S)$ be as in the beginning of §3. For each $\gamma \in \Gamma_K$ there exists a unique matrix $G_\gamma(S) \in M_2^S$ such that*

- (i) $G_\gamma(S) \equiv Id \bmod \pi$;
- (ii) $\Pi_{\hat{A}}(S) \varphi(G_\gamma(S)) = G_\gamma(S) \gamma \Pi_{\hat{A}}(S)$ for all γ .

Proof. Conditions (a) and (b) preceding Proposition 2.6 hold by the discussion in the beginning of §3. Condition (c) preceding Proposition 2.6 and Condition (1) of Proposition 2.6 hold because $\text{Tr}(Q_f) \notin \overline{\mathbb{Q}_p}$ and $\text{Tr}(Q_f^A) \notin \overline{\mathbb{Q}_p}$ respectively, by Lemma 3.1 (i) & (iii). Finally, Condition (d) preceding Proposition 2.6 holds by Lemma 3.1 (iv) with $A = 0$. The proposition follows by Proposition 2.6. \square

For any $\vec{a} \in \mathfrak{m}_E^{[S]}$ and $\dagger \in \{0, \hat{A}\}$, we equip $\mathbb{N}_{\dagger}(\vec{a}) = (\mathcal{O}_E[[\pi]]^{|\tau|}) \eta_1 \oplus (\mathcal{O}_E[[\pi]]^{|\tau|}) \eta_2$ with the φ and Γ_K -actions defined by $(\varphi(\eta_1), \varphi(\eta_2)) = (\eta_1, \eta_2) \Pi_{\dagger}(\vec{a})$ and $(\gamma\eta_1, \gamma\eta_2) = (\eta_1, \eta_2) G_{\gamma, \dagger}(\vec{a})$ respectively.

Corollary 3.3. *The module $\mathbb{N}_{\dagger}(\vec{a})$ with the above φ and Γ_K actions is a Wach module corresponding to some G_K -stable \mathcal{O}_E -lattice of a 2-dimensional crystalline E -representation $V_{\dagger}(\vec{a})$ of G_K with Hodge-Tate weights in $[-k; 0]$.*

Proof. Follows immediately from Proposition 2.8. \square

As in §2.2, the representation $V_{\hat{A}}(\vec{a})$ is independent of the lifting \hat{A} and we simply write $V_A(\vec{a})$.

Lemma 3.4. *If $s \geq k+1$ and $B \in M_2(\mathcal{O}_E[[S]])^{|\tau|}$ is such that $B \equiv Q_f B (p^{f(s-1)} Q_f^{-1}) \bmod I$, then $B \equiv 0 \bmod I$.*

Proof. We may assume that $s-1 = k = k_i$ for all i , otherwise $Q_f B (p^{f(s-1)} Q_f^{-1}) \equiv 0 \bmod I$ and the lemma holds trivially. By Lemma 3.1 (i), $\text{Tr}(Q_f) \notin \overline{\mathbb{Q}_p}$ and [8, Lemma 5.19 & Corollary 5.17] (where in [8] $\overline{Q_f} := Q_f \bmod I$) imply that $Q_f \bmod I = E_{12}$ or E_{21} .

Claim. Let $k_i > 0$ for all i . If $Q_f \bmod I = E_{ij}$ with $i \neq j$ then $p^{fk} Q_f^{-1} \bmod I = -Q_f \bmod I$. If $Q_f \bmod I = E_{11}$, then $p^{fk} Q_f^{-1} \bmod I = E_{22}$, and if $Q_f \bmod I = E_{22}$ then $p^{fk} Q_f^{-1} \bmod I = E_{11}$.

Proof of Claim. By induction on f . For $f = 1$, formula (3.1) becomes

$$P \bmod I = \begin{cases} E_{22} & \text{if } P = t_1, \\ E_{12} & \text{if } P = t_2, \\ E_{11} & \text{if } P = t_3, \\ E_{21} & \text{if } P = t_4, \end{cases}$$

and the claim is clear. Suppose $f \geq 2$. Case (i). $Q_f \bmod I = E_{12}$. If $P_1 P_2 \cdots P_{f-1} \bmod I = E_{11}$ then $P_f \bmod I = E_{12}$. The matrix P_f is of type 2 and by the inductive hypothesis

$$p^{kf} Q_f^{-1} \bmod I = (p^k P_f^{-1}) \cdot \left((p^k P_{f-1}^{-1}) \cdots (p^k P_1^{-1}) \right) \bmod I = -E_{12} \cdot E_{22} = -E_{12}.$$

If $P_1 P_2 \cdots P_{f-1} \bmod I = E_{12}$ then $P_f \bmod I = E_{22}$. The matrix P_f is of type 1 and by the inductive hypothesis

$$p^{kf} Q_f^{-1} \bmod I = (p^k P_f^{-1}) \cdot \left((p^k P_{f-1}^{-1}) \cdots (p^k P_1^{-1}) \right) \bmod I = E_{11} \cdot (-E_{12}) = -E_{12}.$$

The claim follows by the inductive hypothesis, arguing similarly for the other possibilities for $Q_f \bmod I$. \square

The Claim (which applies because we are reduced to the case where $k = k_i > 0$ for all i) and the formula $B \equiv Q_f B (p^{fk} Q_f^{-1}) \bmod I$ imply that $B \equiv -E_{ij} B E_{ij} \bmod I$ with $i \neq j$. From the latter it is immediate that $B \equiv 0 \bmod I$. \square

Proposition 3.5. *Let $A \in M_2(p^{1+\alpha(k-1)} \mathcal{O}_E[[\pi]])^{|\tau|}$ and let $\Pi_{\hat{A}}(\mathcal{S})$ and $G_{\gamma, \hat{A}}(\mathcal{S})$ be as in Proposition 2.6. Then $\Pi_{\hat{A}}(\mathcal{S}) \equiv \Pi(\mathcal{S}) \bmod I$ and $G_{\gamma, \hat{A}}(\mathcal{S}) \equiv G_{\gamma}(\mathcal{S}) \bmod I$.*

Proof. By Lemma 2.5 (iv), $\hat{A} \equiv 0 \bmod I$, hence $\Pi_{\hat{A}}(\mathcal{S}) \equiv \Pi(\mathcal{S}) \bmod I$. Fix a topological generator γ of Γ_K . By the proofs of [8, Propositions 5.9 & 5.11], there exists a matrix $G_{\gamma}^{(k)}(\mathcal{S}) \in M_n^{\mathcal{S}}$ with $G_{\gamma}^{(k)}(\mathcal{S}) \equiv Id \bmod \vec{\pi}$ and a matrix $R^{(k)}(\mathcal{S}) \in M_n^{\mathcal{S}}$ such that

$$(3.2) \quad G_{\gamma}^{(k)}(\mathcal{S}) - \Pi(\mathcal{S}) \cdot \varphi \left(G_{\gamma}^{(k)}(\mathcal{S}) \right) \cdot \gamma \left(\Pi(\mathcal{S})^{-1} \right) = \vec{\pi}^k R^{(k)}(\mathcal{S}).$$

Moreover, by the proof of [8, Lemma 4.1], for all $s \geq k+1$ there exist matrices $G_{\gamma}^{(s)}(\mathcal{S}) \in M_n^{\mathcal{S}}$ and $R^{(s)}(\mathcal{S}) \in M_n^{\mathcal{S}}$ such that $G_{\gamma}^{(s)}(\mathcal{S}) \equiv G_{\gamma}^{(s-1)}(\mathcal{S}) \bmod \vec{\pi}^{s-1} M_n^{\mathcal{S}}$ and

$$(3.3) \quad G_{\gamma}^{(s)}(\mathcal{S}) - \Pi(\mathcal{S}) \cdot \varphi \left(G_{\gamma}^{(s)}(\mathcal{S}) \right) \cdot \gamma \left(\Pi(\mathcal{S})^{-1} \right) = \vec{\pi}^s R^{(s)}(\mathcal{S}).$$

Arguing as in the proof of part (ii) of Proposition 2.6 and taking into account equations (3.2) & (3.3) we see that for all $s \geq k$ there exist matrices $R_{\hat{A}}^{(s)}(\mathcal{S}) \in M_n^{\mathcal{S}}$ such that

$$(3.4) \quad G_{\gamma, \hat{A}}^{(s)}(\mathcal{S}) - \Pi_{\hat{A}}(\mathcal{S}) \cdot \varphi \left(G_{\gamma, \hat{A}}^{(s)}(\mathcal{S}) \right) \cdot \gamma \left(\Pi_{\hat{A}}(\mathcal{S})^{-1} \right) = \vec{\pi}^s R_{\hat{A}}^{(s)}(\mathcal{S})$$

Combining equations (3.2), (3.3) and (3.4) for all $s \geq k$ we write

$$(3.5) \quad G_{\gamma, \dagger}^{(s)}(\mathcal{S}) - \Pi_{\dagger}(\mathcal{S}) \cdot \varphi \left(G_{\gamma, \dagger}^{(s)}(\mathcal{S}) \right) \cdot \gamma \left(\Pi_{\dagger}(\mathcal{S})^{-1} \right) = \vec{\pi}^s R_{\dagger}^{(s)}(\mathcal{S}),$$

with $\dagger \in \{\emptyset, \hat{A}\}$. We defined $G_{\gamma, \hat{A}}^{(k)}(\mathcal{S}) := G_{\gamma}^{(k)}(\mathcal{S})$ and by Lemma 2.10 (ii), $R_{\gamma, \hat{A}}^{(k)}(\mathcal{S}) \equiv R_{\gamma}^{(k)}(\mathcal{S}) \bmod I$. We will show by induction that $G_{\gamma, \hat{A}}^{(s)}(\mathcal{S}) \equiv G_{\gamma}^{(s)}(\mathcal{S}) \bmod I$ and $R_{\hat{A}}^{(s)}(\mathcal{S}) \equiv R^{(s)}(\mathcal{S}) \bmod I$ for all $s \geq k$. For $s \geq k+1$, let $G_{\gamma, \dagger}^{(s)} = G_{\gamma, \dagger}^{(s-1)} + \vec{\pi}^{s-1} H_{\dagger}^{(s)}$, where $H_{\dagger}^{(s)} = H_{\gamma, \dagger}^{(s)} \in M_n(\mathcal{O}_E[[\mathcal{S}]])^{|\tau|}$, and

let $R_{\dagger}^{(s)}(\mathcal{S}) = \overline{R}_{\dagger}^{(s)}(\mathcal{S}) + \pi \cdot C_{\dagger}^{(s)}$ for some matrices $\overline{R}_{\dagger}^{(s)}(\mathcal{S}) \in M_n(\mathcal{O}_E[[\mathcal{S}]])^{|\tau|}$ and $C_{\dagger}^{(s)} \in M_n^{\mathcal{S}}$. By the inductive hypothesis, $\overline{R}_{\hat{A}}^{(s-1)}(\mathcal{S}) + \pi \cdot C_{\hat{A}}^{(s-1)} \equiv \overline{R}_{\hat{A}}^{(s-1)}(\mathcal{S}) + \pi \cdot C_{\hat{A}}^{(s-1)} \pmod{I}$, and since $\overline{R}_{\dagger}^{(s-1)}(\mathcal{S}) \in M_n(\mathcal{O}_E[[\mathcal{S}]])^{|\tau|}$, the latter implies that $\overline{R}_{\hat{A}}^{(s-1)}(\mathcal{S}) \equiv \overline{R}^{(s-1)}(\mathcal{S}) \pmod{I}$. By the proof of [8, Lemma 4.1], the matrices $H_{\dagger}^{(s)}$ can be chosen to be solutions of the equations

$$(3.6) \quad H_{\dagger}^{(s)} - p^{(s-1)} \Pi_{\dagger}^{(0)}(\mathcal{S}) \varphi \left(H_{\dagger}^{(s)} \right) \left(\Pi_{\dagger}^{(0)}(\mathcal{S}) \right)^{-1} = -\overline{R}_{\dagger}^{(s-1)}, \text{ with } \dagger \in \{\emptyset, \hat{A}\}.$$

Let $H_{\dagger}^{(s)} = (H_{1,\dagger}^{(s)}, H_{2,\dagger}^{(s)}, \dots, H_{f-1,\dagger}^{(s)}, H_{0,\dagger}^{(s)})$ and $-\overline{R}_{\dagger}^{(s-1)} = (\overline{R}_{1,\dagger}^{(s-1)}, \overline{R}_{2,\dagger}^{(s-1)}, \dots, \overline{R}_{f-1,\dagger}^{(s-1)}, \overline{R}_{0,\dagger}^{(s-1)})$. Equations (3.6) are equivalent to the systems of equations

$$(3.7) \quad H_{i,\dagger}^{(s)} - P_{i,\dagger} \cdot H_{i+1,\dagger}^{(s)} \cdot (p^{s-1} P_{i,\dagger}^{-1}) = \overline{R}_{i,\dagger}^{(s-1)},$$

for $i = 1, 2, \dots, f$, $\dagger \in \{\emptyset, \hat{A}\}$, and with indices viewed mod f . These imply that

$$(3.8) \quad \begin{aligned} H_{1,\dagger}^{(s)} - Q_{f,\dagger} H_{1,\dagger}^{(s)} (p^{f(s-1)} Q_{f,\dagger}^{-1}) &= \overline{R}_{1,\dagger}^{(s-1)} + Q_{1,\dagger} \overline{R}_{2,\dagger}^{(s-1)} (p^{(s-1)} Q_{1,\dagger}^{-1}) + Q_{2,\dagger} \overline{R}_{3,\dagger}^{(s-1)} (p^{2(s-1)} Q_{2,\dagger}^{-1}) \\ &\quad + \dots + Q_{f-1,\dagger} \overline{R}_{0,\dagger}^{(s-1)} (p^{(s-1)(f-1)} Q_{f-1,\dagger}^{-1}), \end{aligned}$$

where $Q_{i,\dagger} = P_{1,\dagger} \cdots P_{i,\dagger}$ for all $i = 1, 2, \dots, f$. The matrices $H_{i,\dagger}^{(s)}$, $i = 2, 3, \dots, f$ are uniquely determined by the matrix $H_{1,\dagger}^{(s)}$. Let

$$\begin{aligned} V_{\dagger}^{(s)} &= \overline{R}_{1,\dagger}^{(s-1)} + Q_{1,\dagger} \overline{R}_{2,\dagger}^{(s-1)} (p^{(s-1)} Q_{1,\dagger}^{-1}) + Q_{2,\dagger} \overline{R}_{3,\dagger}^{(s-1)} (p^{2(s-1)} Q_{2,\dagger}^{-1}) \\ &\quad + \dots + Q_{f-1,\dagger} \overline{R}_{0,\dagger}^{(s-1)} (p^{(s-1)(f-1)} Q_{f-1,\dagger}^{-1}). \end{aligned}$$

Since $\overline{R}_i^{(s-1)} \equiv \overline{R}_{i,\hat{A}}^{(s-1)} \pmod{I}$, and since $Q_i \equiv Q_{i,\hat{A}} \pmod{I}$ and $p^{i(s-1)} Q_{i,\hat{A}}^{-1} \equiv p^{i(s-1)} Q_i^{-1} \pmod{I}$ for all i , it follows that $V_{\dagger}^{(s)} \equiv V_{\hat{A}}^{(s)} \pmod{I}$. Since since $Q_{1,\hat{A}} \equiv Q_1 \pmod{I}$, the latter and equations (3.8) imply that

$$H_{1,\hat{A}}^{(s)} - H_1^{(s)} = Q_f \left(H_{1,\hat{A}}^{(s)} - H_1^{(s)} \right) \left(p^{f(s-1)} Q_f^{-1} \right) \pmod{I},$$

and Lemma 3.4 applied for $B = H_{1,\hat{A}}^{(s)} - H_1^{(s)}$ implies that $H_{1,\hat{A}}^{(s)} \equiv H_1^{(s)} \pmod{I}$. Since $P_{i,\hat{A}} \equiv P_i \pmod{I}$ and $R_{i,\hat{A}}^{(s-1)} \equiv R_i^{(s-1)} \pmod{I}$, equations (3.7) imply that $H_{\hat{A}}^{(s)} \equiv H^{(s)} \pmod{I}$. Since $G_{\gamma,\dagger}^{(s)} = G_{\gamma,\dagger}^{(s-1)} + \pi^{s-1} H_{\dagger}^{(s)}$, the inductive hypothesis implies that $G_{\gamma,\hat{A}}^{(s)}(\mathcal{S}) \equiv G_{\gamma}^{(s)}(\mathcal{S}) \pmod{I}$. Formula (3.5) yields $\pi^s \cdot \gamma \Pi_{\hat{A}}(\mathcal{S}) \cdot R_{\hat{A}}^{(s)} \equiv \pi^s \cdot \gamma \Pi(\mathcal{S}) \cdot R^{(s)} \pmod{I}$. Since the coordinate matrices of both the matrices $\gamma \Pi_{\hat{A}}(\mathcal{S})$ and $\gamma \Pi(\mathcal{S})$ coincide mod I and have nonzero determinants mod I , it follows that $R_{\hat{A}}^{(s)} \equiv R^{(s)} \pmod{I}$, and this finishes the induction. We have shown that $G_{\gamma,\hat{A}}^{(s)}(\mathcal{S}) \equiv G_{\gamma}^{(s)}(\mathcal{S}) \pmod{I}$ for any $s \geq k$. Since $G_{\gamma,\dagger}(\mathcal{S}) = \lim_{s \rightarrow \infty} G_{\gamma,\dagger}^{(s)}(\mathcal{S})$ it follows that $G_{\gamma,\hat{A}}(\mathcal{S}) \equiv G_{\gamma}(\mathcal{S}) \pmod{I}$, and this finishes the proof. \square

Corollary 3.6. *Let $A \in M_2(p^{1+\alpha(k-1)} \mathcal{O}_E[[\pi]])^{|\tau|}$. Then for any $\vec{a} \in \mathfrak{m}_E^f$ there exist G_{K_f} -stable \mathcal{O}_E -lattices with respect to which $\overline{V}_A(\vec{a}) \simeq \overline{V}_0(\vec{0})$.*

Proof. Proposition 3.5 implies that $\Pi_{\hat{A}}(\vec{a}) \equiv \Pi(\vec{0}) \pmod{\mathfrak{m}_E}$ and $G_{\gamma,\hat{A}}(\vec{a}) \equiv G_{\gamma}(\vec{0}) \pmod{\mathfrak{m}_E}$ for all $\gamma \in \Gamma_K$. The rest of the proof is identical to that of [8, Theorem 4.7]. \square

Parts (ii) and (iii) of Theorem A follow from Proposition 2.9 and Corollary 3.6. The following proposition proves part (i) and finishes the proof of the theorem.

Proposition 3.7. *Let $A \in M_2(p^{\alpha(k-1)}\mathcal{O}_E[[\pi]])^{|\tau|}$ and $\vec{\alpha} \in (p^m \mathfrak{m}_E)^f$. We define the rank two filtered φ -module $(\mathbb{D}_A(\vec{\alpha}), \varphi)$ with Frobenius endomorphism*

$$(3.9) \quad (\varphi(\eta_1), \varphi(\eta_2)) := (\eta_1, \eta_2) P_A(\vec{\alpha})$$

and filtration

$$\mathrm{Fil}^j(\mathbb{D}_A(\vec{\alpha})) := \begin{cases} E^{|\tau|}\eta_1 \oplus E^{|\tau|}\eta_2 & \text{if } j \leq 0, \\ E^{|\tau_{I_0}|}(\vec{x}\eta_1 + \vec{y}\eta_2) & \text{if } 1 \leq j \leq w_0, \\ E^{|\tau_{I_1}|}(\vec{x}\eta_1 + \vec{y}\eta_2) & \text{if } 1 + w_0 \leq j \leq w_1, \\ \dots\dots\dots & \\ E^{|\tau_{I_{t-1}}|}(\vec{x}\eta_1 + \vec{y}\eta_2) & \text{if } 1 + w_{t-2} \leq j \leq w_{t-1}, \\ 0 & \text{if } j \geq 1 + w_{t-1}, \end{cases}$$

with

$$(3.10) \quad (x_i, y_i) = \begin{cases} (1, -\alpha_i) & \text{if } P_i \text{ has type 1 or 2,} \\ (-\alpha_i, 1) & \text{if } P_i \text{ has type 3 or 4.} \end{cases}$$

The filtered φ -modules $\mathbb{D}_A(\vec{\alpha})$ are weakly admissible and correspond to 2-dimensional crystalline E -linear representations $V_A(\vec{\alpha})$ of G_{K_f} with Hodge-Tate type $\mathrm{HT}(\tau_i) = \{0, -k_i\}$.

Proof. We compute the filtered φ -modules given rise to by the Wach modules $\mathbb{N}_{\hat{A}}(\vec{a})$. The φ -action is obviously given by formula (3.9), and it suffices to compute $\mathrm{Fil}^j(\mathbb{N}_{\hat{A}}(\vec{a})/\pi\mathbb{N}_{\hat{A}}(\vec{a}))$. By Theorem 2.4, $\vec{x}\eta_1 + \vec{y}\eta_2 \in \mathrm{Fil}^j(\mathbb{N}_{\hat{A}}(\vec{a}))$ if and only if there $\varphi(\vec{x}\eta_1 + \vec{y}\eta_2) \in q^j\mathbb{N}_{\hat{A}}(\vec{a})$. Written in matrix form and recalling that the φ -action on $\mathcal{O}_E[[\pi]]^{|\tau|}$ is given by formula (2.1), the latter is equivalent to the existence of vectors $\vec{u}_1, \vec{u}_2 \in \mathcal{O}_E[[\pi]]^{|\tau|}$ such that

$$(3.11) \quad (e_{i-1}\eta_1, e_{i-1}\eta_2) \left(Id + \hat{A}_i(\vec{a}) \right) \Pi_i(a_i) (\varphi(x_i), \varphi(y_i))^t = (q^j e_{i-1}\eta_1, q^j e_{i-1}\eta_2) (u_1^{i-1}, u_2^{i-1})^t$$

for all $i = 0, 1, \dots, f-1$, where e_i are the idempotents of $\mathcal{O}_E[[\pi]]^{|\tau|}$. Let

$$(\zeta_1^{i-1}, \zeta_2^{i-1}) := (e_{i-1}\eta_1, e_{i-1}\eta_2) \left(Id + \hat{A}_i(\vec{a}) \right) \text{ and } (v_1^{i-1}, v_2^{i-1})^t := \left(Id + \hat{A}_i(\vec{a}) \right)^{-1} (u_1^{i-1}, u_2^{i-1})^t.$$

Since $\hat{A} \equiv A \bmod \pi$ and $\det(Id + A_i) \in \mathcal{O}_E^\times$ it follows that $Id + \hat{A}_i(\vec{a}) \in \mathrm{GL}_2(\mathcal{O}_E[[\pi]])$ for all i . Therefore (ζ_1^i, ζ_2^i) is an ordered basis of $e_i\mathbb{N}_{\hat{A}}(\vec{a})$ for all i and equation (3.11) implies

$$(\zeta_1^{i-1}, \zeta_2^{i-1}) \Pi_i(a_i) (\varphi(x_i), \varphi(y_i))^t = (q^j \zeta_1^{i-1}, q^j \zeta_2^{i-1}) (v_1^{i-1}, v_2^{i-1})^t.$$

Assume that $\Pi_i(X_i)$ is of type 2. Arguing as in the proof of [8, Proposition 5.22] we see that $x_i, y_i \equiv 0 \bmod \pi$ if $j \geq k_i$ and $\pi^j \mid x_i + y_i a_i z_i$ for $1 \leq j \leq k_i$. Since $z_i \bmod \pi = p^m$ and $\alpha_i := p^m a_i$,

$$e_i \vec{x}\eta_1 + e_i \vec{y}\eta_2 + \pi \mathbb{N}_{\hat{A}}(\vec{a}) = \begin{cases} \alpha_i \bar{y}_i e_i \eta_1 + \bar{y}_i e_i \eta_2 + \pi \mathbb{N}_{\hat{A}}(\vec{a}) & \text{if } 1 \leq j \leq k_i, \\ 0 & \text{if } j \geq k_i \end{cases}$$

where $\bar{y}_i = y_i \bmod \pi$ can be any element of \mathcal{O}_E . Hence

$$e_i \mathrm{Fil}^j(\mathbb{N}_{\hat{A}}(\vec{a})/\pi\mathbb{N}_{\hat{A}}(\vec{a})) = \begin{cases} e_i(\mathcal{O}_E^{|\tau|})\eta_1 \oplus e_i(\mathcal{O}_E^{|\tau|})\eta_2 & \text{if } j \leq 0, \\ e_i(\mathcal{O}_E^{|\tau|})(\vec{x}\eta_1 + \vec{y}\eta_2) & \text{if } 1 \leq j \leq k_i, \\ 0 & \text{if } j \geq 1 + k_i, \end{cases}$$

with $(x_i, y_i) = (-\alpha_i, 1)$. Computing for the other choices of $\Pi_i(a_i)$, we see that for all $i \in I_0$, (x_i, y_i) is as in formula (3.10) and the proof follows as in [8, Proposition 5.22]. To finish the proof, notice that by the definition of the polynomials z_i appearing in the matrices Π_i , the sets $\{(z_1 a_1 \bmod \pi, \dots, z_0 a_0 \bmod \pi)\}$, where $(a_1, \dots, a_f) \in \mathfrak{m}_E^f$ and $(p^m \mathfrak{m}_E)^f$ coincide. We let $\vec{\alpha} := p^m \cdot \vec{a}$ for any vector $\vec{a} \in \mathfrak{m}_E^f$ and parametrize our families by the vectors $\vec{\alpha}$. Finally, by Theorem 2.4,

$$\mathbb{D}_{\text{cris}}(V_A(\vec{\alpha})) \cong E^{|\tau|} \otimes_{\mathcal{O}_{E^{|\tau|}}} \text{Fil}^j(\mathbb{N}_{\hat{A}}(\vec{a})/\pi \mathbb{N}_{\hat{A}}(\vec{a})).$$

Since $\mathbb{D}_A(\vec{\alpha}) = E^{|\tau|} \otimes_{\mathcal{O}_{E^{|\tau|}}} \text{Fil}^j(\mathbb{N}_{\hat{A}}(\vec{a})/\pi \mathbb{N}_{\hat{A}}(\vec{a}))$, the filtered φ -modules $\mathbb{D}_A(\vec{\alpha})$ are weakly admissible. \square

Remark 3.8. For fixed \vec{a} the filtered φ -modules $\mathbb{N}_{\hat{A}}(\vec{a})/\pi \mathbb{N}_{\hat{A}}(\vec{a})$ only depend on $A \equiv \hat{A} \bmod \pi$. This is clear from the proof of Proposition 3.7 (essentially by Theorem 2.4).

REFERENCES

- [1] Berger, L., Limites de représentations cristallines, *Comp. Math.* 140, (2004) 1473-1498.
- [2] Berger, L., Local constancy for the reduction mod p of 2-dimensional crystalline representations. *Bull. London Math. Soc.* Advance Access published November 7, 2011.
- [3] Berger, L., Breuil, C., Towards a p -adic Langlands programme. Summer school on p -adic arithmetic geometry in Hangzhou. <http://perso.ens-lyon.fr/laurent.berger/autrestextes/hangzhou.pdf>
- [4] Berger, L., Li, H., Zhu, H.J., Construction of some families of 2-dimensional crystalline representations. *Math. Ann.* 329, (2004) 365-377.
- [5] Buzzard, K., Diamond, F., Jarvis, F., On Serre's conjecture for mod l Galois representations over totally real fields. *Duke Math. J.* 155 (2010), no. 1, 105-161.
- [6] Colmez, P., Fontaine, J.-M., Construction des représentations p -adiques semi-stables. *Invent. Math.* 140, (2000) 1-43.
- [7] Dousmanis, G., Rank two filtered (φ, N) -modules with Galois descent data and coefficients. *Trans. Amer. Math. Soc.* 362 (2010), no. 7, 3883-3910.
- [8] Dousmanis, G., On reductions of families of crystalline Galois representations. *Doc. Math.* 15 (2010), 873-938.
- [9] Fontaine, J.-M., Le corps des périodes p -adiques. *Périodes p -adiques* (Bures-sur-Yvette, 1988). *Asterisque* 223, (1994) 59-111.
- [10] Fontaine, J.-M., Représentations p -adiques des corps locaux I. *The Grothendieck Festschrift, Vol II*, (1990) 249-309, *Prog. Math.* 87, Birkhäuser Boston, Boston, MA.